INSTANTANEOUS LIABILITY RULE AUCTIONS: THE CONTINUOUS EXTENSION OF HIGHER-ORDER LIABILITY RULES

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ABSTRACT

A "higher-order" liability regime—in which a plaintiff and a defendant have a sequence of alternating options to take a disputed entitlement—can enhance allocative efficiency by harnessing the private information possessed by both litigants. Indeed, infinite order liability regimes can, as a theoretical matter, assure first-best efficiency. Such iterated taking regimes have, however, been criticized as (i) generating excessive (and debilitating) taking costs, and (ii) being infeasible with regard to intangible entitlements. This Paper shows that courts can replicate the first-best efficiency of infinite-stage liability rule via an instantaneous auction mechanism. This instantaneous mechanism avoids the excessive taking cost criticism (because the disputants merely submit a single report of value). Unlike many auctions, the mechanism also allows courts to pursue equitable goals by dividing the bulk of expected gains to either plaintiff or defendant (without undermining the first-best allocative efficiency). A derivation is given of the explicit formula for the basic element of the procedure, which we call the "damage curve" and which determines the amount of damages that the winner of the auction must pay the loser. This formula holds for arbitrary joint probability distributions of the valuations of the asset, whether correlated or uncorrelated. Explicit damage curves are calculated for several concrete examples, illustrating both correlated and uncorrelated cases.

I. Introductory remarks and basic concepts

Kaplow and Shavell[4]—formalizing Calabresi and Melamed[5]—showed that "liability rules" can harness the private information of a potential taker to enhance allocative efficiency. For example,

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[4]Louis Kaplow and Steven Shavell, *Property Rules Versus Liability Rules: An Economic Analysis*, 109 Harv. L. Rev. 713 (1996).

[5]Guido Calabresi and Douglas Melamed, *Property Rules, Liability Rules, and Inalienability: One View of the Cathedral*, 85 Harv. L. Rev. 1089 (1972).

when nuisance damages are set at the expected value of the pollutee (takee) a potential polluter (taker) will be induced to take only if she expects that the taking would enhance efficiency.

But while simple liability rules can do a better job at economizing on the private information of potential takers than property rules can, Ayres and Balkin[6] pointed out that they fail to harness the private information of the other side of the dispute. Ayres and Goldbart[7]showed that that giving the disputants a sequence of alternating options to take a disputed entitlement at successively increasing court ordered damages could even better enhance allocative efficiency by harnessing the private information of both parties. This "higher order" liability rule resembled an auction in which each successive taking amounted to a bid signaling a higher private value. And indeed, a potentially infinite sequence of takings could, in theory, just like a traditional auction, produce first-best allocative efficiency—with the disputed entitlement always being allocated to the higher valuer.

But, unlike traditional auctions, where the winning bidder pays a third party (the seller), this regime represented an internal auction, in which the winning bidder paid the losing bidder. Ayres and Goldbart[8] showed how this internal auction feature enhanced the distributional flexibility of courts to respect the equitable claims of the polutee or the polluter (or enhance *ex ante* investment incentives). Indeed, it is possible to construct a higher-order liability rule so as to maintain first-best allocative efficiency and divide the expected value of the entitlement between the disputants as the court sees fit.

While infinite staged higher-order liability rules thus have attractive theoretical properties, they have been criticized as being impractical. Kaplow and Shavell[9] point out that iterated taking regimes might (i) generate excessive (and debilitating) taking costs and (ii) be infeasible with regard to intangible entitlements.

The present article advances the ball by showing that it is possible to implement an infinite-stage liability rule with an instantaneous procedure that avoids the takings problems identified by Kaplow and Shavell. Moreover, our procedure achieves first-best efficiency in a model without any possibility of consensual trade. Ours is a direct mechanism in which the disputants are asked to report how much they value the entitlement to the court, with knowledge that the court will (i) allocate the entitlement to the disputant submitting the higher report and (ii) assess damages according to a pre-specified damage curve (that is a function of both disputant's reports).

We show that there exists an equilibrium in which disputants report their true values and the

[6]I. Ayres and J.M. Balkin, *Legal Entitlements as Auctions: Property Rules, Liability Rules, and Beyond*, 106 Yale L.J. 703, 729-33 (1996).

[7]I. Ayres and P.M. Goldbart, *Optimal Delegation and Decoupling in the Design of Liability Rules*, 100 Mich. L. Rev., 1-79 (2001).

[8]See *supra* note 7.

[9]See *supra* note 4.

entitlement is accordingly allocated by the court to the first-best valuer. We present a derivation of the explicit formula for the core feature of the procedure, what we call the "damage curve", which determines the amount of damages that the winner of the auction must pay the loser. This formula holds for arbitrary joint probability distributions of the valuations of the asset, whether correlated or uncorrelated. We calculate explicit damage curves for several concrete (correlated and uncorrelated) examples.

There are of course other auction mechanisms that could also achieve first-best allocative efficiency. For example, the government could (i) exercise its eminent domain option over the disputed entitlement (paying the *ex ante* expected value of the entitlement to one of the disputants), and then (ii) auction the entitlement to the highest bidder (with the government keeping the revenues of the auction). But the government knows more about the value of the entitlement at the end of the auction than at the beginning, because the very process of bidding reveals information about the disputants' private valuations. Accordingly, an advantage of our proposal over other auction mechanisms is that it allows more nuanced divisions of the entitlement's ultimate value between the disputants.

A. Review of liability rules

The aim of the present subsection is to introduce the reader to the "taxonomy" of liability rules: the traditional ones, as well as those proposed in the literature prior to the present work. Although the liability rules that we shall review in the present section certainly improve upon the allocative efficiency of the property rules, they do not yield perfect efficiency. The issue of how courts may apply judicial rulings that yield perfect efficiency by harnessing privately held information—the central mission of this Paper—is discussed in section III. The analysis of efficiency of liability rules reviewed in the present section is beyond the scope of the present Paper. The interested reader is referred to a companion Paper,[10] as well as prior work of Ayres and Goldbart.[11]

1. Traditional property and liability rules

We start off by examining the traditional property and liability rules. The insightful observation of Calabresi and Melamed was that the rules can be categorized via (a) the holder of the initial entitlement to the asset (the plaintiff or the defendant), and (b) the type of rule (property or liability). We now list their specific examples:

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[10] S. Knysh, P.M. Goldbart and I. Ayres, *Design of Efficient Legal Liability Rules: Comparison of Discrete Vanilla and Exotic Variants.*

[11] See *supra* note 7.

Injunctive relief (Rule 1) The plaintiff gets the initial entitlement to the asset; the rule is of the property type.

Injunction denied (Rule 3) The defendant gets the initial entitlement to the asset; the rule is of the property type.

Compensatory damages (Rule 2) The plaintiff gets the initial entitlement but the defendant receives a call option—a right to acquire the asset by paying damages. The rule is of the liability (call) type.

Rule 4 The defendant gets the initial entitlement but the plaintiff receives a call option. The rule is of the liability (call) type.[12]

The original categorization of Calabresi and Melamed is augmented with another type of rule, in which the forced sale (a call option) is replaced with a forced purchase (a put option). The choice of terminology again comes from finance. We summarize these rules in a 2x3 extended Calabresi-Melamed Table.[13]

Initial Entitlement	*Property Rule*	*Liability Rule (Call)*	*Liability Rule (Put)*
Plaintiff	Rule 1	Rule 2	Rule 6
Defendant	Rule 3	Rule 4	Rule 5

Table 1: Extended Calabresi-Melamed table of rules.

As a final example, we mention Rule 6, which grants the plaintiff both the initial entitlement and the put option—the right to demand damages instead of the asset. Rule 5 is unlikely to be used, except potentially in a coming to the nuisance setting. It is, nonetheless, included for completeness.

In another invocation of the standard financial terminology, we shall call Rules 1-6 "vanilla" rules (cf. futures, call and put options in finance); the other, more complicated, types of rules shall be termed "exotic." Having dealt with the vanilla rules we move on to discuss their exotic variants.

2. Exotic liability rules

In the present section we shall list the exotic rules proposed elsewhere, and give a brief description of them.[14]

[12] Spur Indus., Inc. v. Del E. Webb Dev. Co., 494 P.2d 700, 708 (Ariz. 1972).

[13] Ian Ayres, The 1998 Monsanto Lecture: *Protecting Property with Puts*, 32 Val. U. Rev. 793 (1998).

[14] See *supra* note 7 and *supra* note 10.

Double Call The plaintiff gets the initial entitlement. The defendant gets a call option—the right to buy the asset for D_Δ. The plaintiff can prevent this by paying D_Π, which can be thought of as the plaintiff having a call-back option. A second type of double-call rule is obtained by interchanging the roles of plaintiff and defendant. Note that the last chooser—the plaintiff—will exercise his option if $V_\Pi > Q_\Pi = D_\Delta + D_\Pi$. However, the first chooser—the defendant—must solve a slightly more complicated problem in order to determine whether to exercise his option. The pivot value Q_Δ (i.e. the amount such that the defendant exercises his option only if $V_\Delta > Q_\Delta$) is somewhat smaller than the damages: $Q_\Delta < D_\Delta$. This amounts to *strategic underbidding*, i.e., the exercising of the option for a loss, in the hope of reaping higher rewards, should the opponent exercise the call-back option.

Double Put The plaintiff gets the initial entitlement and a put option—the right to sell the asset for D_Π. The defendant can prevent the forced transfer by paying $D_\Delta < D_\Pi$, which can be thought of as a put-back option.

Joint Veto The asset stays with the initial asset holder, with no damages being paid unless both litigants agree to transfer the asset for damages D. This rule can be viewed in two equivalent ways: as a special case of either (a) the Double-Call rule, or (b) the Double-Put rule, in both cases the second damages (i.e. the damages paid to prevent transfer) being set to zero. Note that no strategic underbidding takes place in this case.

3. Generalization to higher-order rules

This section describes and generalizes the notion of higher-order liability rules—which represent a sequence of alternating taking options (iterated call rules) or alternating giving options (iterated put rules).

Iterated Call Imagine that one of the disputants (without loss of generality) called "plaintiff" gets the initial entitlement. In the first stage, the defendant gets the initial option of buying the asset for D_Δ^1. The plaintiff gets the option of preventing this transfer by paying D_Π^1. In the second stage, the defendant has the option of responding by increasing his bid to D_Δ^2, in response to which the plaintiff can again prevent transfer by increasing his offer to D_Π^2. This process continues for up to n stages, the n-stage game being fully described by the two sequences of damages:

$$D_\Delta^1 < D_\Delta^2 < \cdots < D_\Delta^n \,, \tag{I.1a}$$

$$D_\Pi^1 < D_\Pi^2 < \cdots < D_\Pi^n \,. \tag{I.1b}$$

Iterated Put An alternative higher-order liability rule entails a potential sequence of being given put options. Again imagine that the plaintiff get the initial entitlement. The plaintiff also gets the initial option to forcefully transfer the asset to the defendant and to receive damages D^1_Π. The defendant can prevent transfer by paying D^1_Δ. In response, the plaintiff can lower the damages he is to receive to D^2_Π, and the defendant can, in turn, prevent transfer by paying D^2_Δ. The n-stage game is fully described by the two sequences of damages

$$D^1_\Pi > D^2_\Pi > \cdots > D^n_\Pi \,, \tag{I.2a}$$
$$D^1_\Delta < D^2_\Delta < \cdots < D^n_\Delta \,. \tag{I.2b}$$

Having reviewed the vanilla and exotic options that have been explored to date, we now turn to the main focuses of the present Paper, viz., the development of more powerful liability rules and an analysis of them in game-theoretic terms.

II. Extension to continuous rules

The purpose of this section is to explain in detail a family of liability rules that are yet more exotic than those considered in the previous section and that have the important virtues of (a) containing the rules of the previous section as simple special cases, and (b) being capable, by appropriate tuning of the parameters, of achieving perfect efficiency. In addition to formulating these rules, we shall explore a number of illustrative examples. In the following section we shall examine the new rules from a game-theoretic perspective; and in the concluding section of the Paper, we shall, among other things, draw analogies between these new rules and the familiar topic of auctions.

A. Continuous call rule

Let us examine more closely the *iterated call* rule. It will prove convenient to make a slight change of language here. We wish to refer to the exercising of an option as the *making of a bid*, and to the corresponding damages as the *amount of the bid.* Two factors distinguish this notion of bidding from the conventional one. First, the permitted bid amounts are drawn from a discrete set, which is specified by the court. And second, whereas in conventional bidding *any* bid must exceed the *opponent's* previous bid, in the present setting a litigant's bid must exceed only his *own* previous one. Stated mathematically, the set of interlaced conditions

$$0 < D^1_\Delta < D^1_\Pi < D^2_\Delta < D^2_\Pi < \cdots < D^n_\Delta < D^n_\Pi \,, \tag{II.1}$$

is replaced by the two separate sets of conditions[15]

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[15]We simply rewrite Eqs. (I.1a) and (I.1b).

$$0 < D^1_\Delta < D^2_\Delta < \cdots < D^n_\Delta\,, \tag{II.2a}$$
$$0 < D^1_\Pi < D^2_\Pi < \cdots < D^n_\Pi\,. \tag{II.2b}$$

Let us now make a natural generalization of this scheme. We take the limit $n \to \infty$, so that the increments of the bid amounts, $\{D^{k+1}_\Pi - D^k_\Pi\}$ and $\{D^{k+1}_\Delta - D^k_\Delta\}$, become infinitesimal quantities. It is convenient (and always possible) to view the damages $\{D^k_\Pi\}$ and $\{D^k_\Delta\}$ as the values of a pair of continuous *damages functions* $D_\Pi(s)$ and $D_\Delta(s)$, evaluated at the arguments $\{s^k \equiv (k-1)/n\}$. In order to meet the conditions that the pair of discrete sequences of damages $\{D^k_\Pi\}$ and $\{D^k_\Delta\}$ each be strictly monotonically increasing, we shall require that the derivatives of the damages functions obey $D'_\Pi(s) > 0$ and $D'_\Delta(s) > 0$, where the prime indicates a derivative.

As another slight change of language, we shall refer to the *arguments* of the damages functions $\{s^k\}$ as the *bids*, rather than the actual values of the damages functions at these arguments. We shall refer to the latter as *bid amounts*. As, in the continuous (i.e. $n \to \infty$) limit, the bid increments $s^{k+1} - s^k$ $(= 1/n)$ tend to zero, the bids are drawn from the continuous interval $[0;1)$. The convenience of this focus on the bid s [rather than the bid amounts $D_\Pi(s)$ and $D_\Delta(s)$] lies in the intuitiveness of the notion that it is the highest bid that wins (i.e. ends up with the asset). To emphasize this point, consider a situation in which a plaintiff and a defendant respectively make bids s_Π and s_Δ, with $s_\Pi > s_\Delta$. Then the plaintiff's bid is the winning bid, even though it perfectly well may happen that $D_\Pi(s_\Pi) < D_\Delta(s_\Delta)$.

In setting up the continuum generalization of the iterated call rule, we shall be introducing three conceptual steps. First, we propose that the procedure in which the explicit bids and counter-bids are made is replaced by one in which the bids (still drawn from the discrete set $\{s^k\}$) are announced by the court. Second, noting that bid increments tend to zero, we may assume that the current bid, represented by the number s, increases from zero to one, continuously with time. Third, we entirely eliminate the explicit bidding involved, by requiring the parties simply to secretly submit their final bids to the court.

In the scheme that follows from making the first step, the bids are announced sequentially, one by one, starting with the smallest bid of zero. At each step, there are three possible outcomes: (i) the defendant folds [i.e. allows the plaintiff to control the asset in exchange for the previously announced damages $D_\Pi(s^{k-1})$][16]; (ii) the defendant stays but the plaintiff folds [i.e. allows the defendant to control the asset in exchange for the damages $D_\Delta(s^k)$]; (iii) both defendant and plaintiff stay [i.e. that, to control the asset, the defendant is willing to pay the bid amount $D_\Delta(s^k)$ and the plaintiff is willing to pay the bid amount $D_\Pi(s^k)$], in which case the next bid is announced. If either (i) or (ii) is realized then the procedure stops, the asset is transferred accordingly, and the

[16] We may take $D_\Pi(s^0) = 0$, which means that the asset simply stays with the plaintiff if the defendant chooses not to exercise his first option.

appropriate damages are paid.

Upon making the second step, in which we imagine letting the bid rise from zero to one, continuously in time, the litigants need only to indicate, at some time, their desire to fold. In the (unlikely) event of the litigants folding simultaneously, priority is given to the defendant (i.e. the defendant is taken to be the party who has folded first).

In the third, and final, step we observe that neither party has received any useful information until the bidding ends with one or other party being the winner. Thus, the parties know their maximum bids before bidding has commenced. And, thus, the entire bidding procedure can be dispensed with in favor of a procedure in which the parties simply furnish the court with their maximum bids. The court can use this information to immediately ascertain which party is the winner, as well as the amount of the winning bid[17]. The bidding stops when s reaches the smaller of the two secret bids, s_Π and s_Δ. Therefore, the party that made the highest bid is the winner. The damages are determined, however, by the *loser's* bid, as this is the bid at which the bidding stops.

It remains to mention that the restriction that s lie in the interval $[0;1)$ is quite arbitrary: the essentials of the problem are invariant with respect to arbitrary reparametrization. That is, for any monotonically increasing function of s, say $t(s)$, one equivalently can work with functions $\widetilde{D}_\Pi(t) \equiv D_\Pi(t(s))$ and $\widetilde{D}_\Delta(t) \equiv D_\Delta(t(s))$. In the resulting scheme, the players should choose their bids to be $t_\Pi = t(s_\Pi)$ and $t_\Delta = t(s_\Delta)$, where s_Π and s_Δ would be their bids under the old scheme. As $t(s)$ is monotonically increasing, the winner remains the same; the amount of damages paid, as determined by the lower bid, would similarly be unchanged. If one takes, for instance, $t(s) = t/(1-t)$, the domain of damages functions would be mapped from the interval $[0;1)$ into the entire positive real axis $[0;+\infty)$.

The final version of the bidding procedure admits an appealing geometric interpretation, which we now explain. The damages functions $D_\Pi(s)$ and $D_\Delta(s)$ define, parametrically, a curve in the (D_Π, D_Δ) plane (see figure 1). The positivity of $D'_\Pi(s)$ and $D'_\Delta(s)$ implies that the tangent to the curve always points towards the positive quadrant (i.e. angles from $0°$ to $90°$). The bids correspond to points on this curve, the higher bid being the point lying further along the curve (in the direction of increasing damages). The damages are determined by projecting the point for the losing bid on to the D_Π axis (if the plaintiff is the winner) or on to the D_Δ axis (if the defendant is the winner). This geometric reinterpretation illuminates the reparametrization freedom: the pairs $\{D_\Pi(s), D_\Delta(s)\}$ and $\{\widetilde{D}_\Pi(t), \widetilde{D}_\Delta(t)\}$ define the same curve.

[17]A popular web auction eBay(tm) has a system called *proxy bidding,* which lets users indicate their maximum bid (which is kept private), and simulates the bidding process by bidding incrementally on behalf of each user up to his maximum bid.

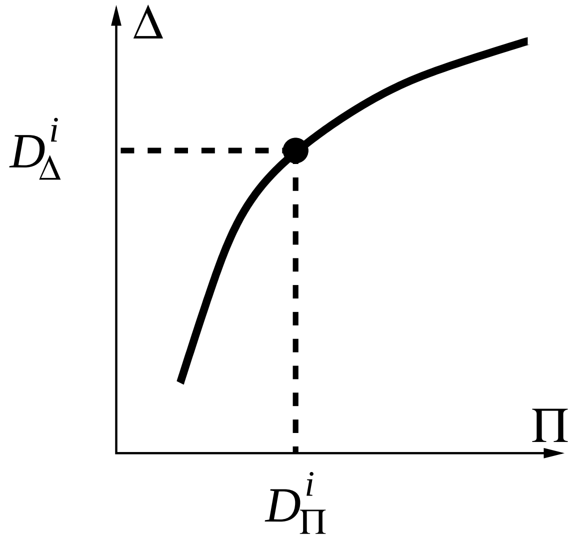

Fig. 1.— An example of the parametrically defined bidding curve. The damages paid are obtained by projecting the lower bid onto Π- or Δ-axis.

B. Continuous put rule

We now examine the continuous put rule. Recall that the iterated put is characterized by the sets of damages $D^1_\Pi > D^2_\Pi > \cdots > D^n_\Pi$ and $D^1_\Delta < D^2_\Delta < \cdots < D^n_\Delta$.[18]We proceed to define damages functions $D_\Pi(s)$ and $D_\Delta(s)$ that satisfy $D_\Pi(s^k) = -D^k_\Delta$ and $D_\Delta(s^k) = D^k_\Pi$. This definition may seem peculiar, however it represents the fact that whoever ends up with the asset pays damages $D_\alpha(s)$, where α stands for either Π or Δ. Indeed, if the plaintiff exercises his option, the defendant (Δ) ends up with the asset and pays $D^k_\Pi \equiv D_\Delta(s^k)$. If the defendant exercises his put-back option, the plaintiff receives the asset and D^k_Δ in damages (or, equivalently, pays the negative amount $-D^k_\Delta = D_\Pi(s^k)$ in damages). For the continuous put rule we prefer to define $s^k \equiv (n-1-k)/n$, so that one still has $D'_\Pi(s) > 0$ and $D'_\Delta(s) > 0$, but now, during the bidding, s *decreases* from 1 to 0. We now consider the last (and somewhat tricky) step: in contrast to the continuous put case, the asset ends up in the hands of the party who *refused* to continue the bidding, i.e., did not exercise his put option.

In constructing the generalized continuous put rule, we again make three revisions of the bidding game. In the first revision, the bids are announced by the court, and the litigants announce whether they are willing to exercise their put options. Should one of them refuse, he becomes the owner of the asset and pays damages determined by the current bid. If both refuse, priority is given to the initial asset holder (the plaintiff). The second revision lets the bid decrease continuously from 1 to 0, until one of the litigants announces his intention to *take* the asset. The damages are

[18]See Eqs. (I.2a) and (I.2b).

determined by the current bid. In the final revision, the litigants submit their secret *taking bids*. The court then assigns the asset to the party with the higher bid. The damages are determined by the *winning* bid, unlike the situation in continuous call rule, in which the losing bid is used to calculate the damages. The difference between the continuous call and the continuous put rules is that the former favors strategic overbidding whereas the latter favors strategic underbidding. The continuous put also admits a geometric interpretation: the winning bid (the point further along the curve) determines the winner, and its projection onto either the D_Π or the D_Δ axis determines the damages exchanged. Similarly to the continuous call case, a reparametrization can be used to relax the condition that s lie in the interval $[0;1)$. Note that it is more convenient and slightly more general to relax another constraint—that $D_\Pi(s) < 0$.

C. Determining optimal strategies

The model that we consider throughout the Paper is the familiar one in which the plaintiff and defendant each possess an element of private information—the value that they place on a right (or *asset*, as we shall commonly refer to it) that is under dispute. These private valuations are denoted V_Π and V_Δ for the plaintiff and defendant, respectively; we assume that they are random variables[19] distributed according to a joint probability distribution (j.p.d.) which we denote $f(V_\Pi, V_\Delta)$. This distribution is assumed to be public knowledge:[20] it is known to the plaintiff, to the defendant, and to the court. We shall say that the private valuations are uncorrelated if the j.p.d. is factorizable, i.e., if $f(V_\Pi, V_\Delta) = f_\Pi(V_\Pi) f_\Delta(V_\Delta)$. Otherwise, we shall say that the private valuations are correlated.

For the correlated case, we shall also make use of the conditional probability densities: $f_\Pi(V_\Pi|V_\Delta) \equiv f_\Pi(V_\Pi, V_\Delta)/\int dx\, f_\Pi(x, V_\Delta)$ and similarly for $f_\Delta(V_\Delta|V_\Pi)$.

1. *Continuous call rule*

In the present section we assume that s_Π and s_Δ are, respectively, the bids submitted by the plaintiff and the defendant (referred to in this section as the players Π and Δ). In addition to conditional probability distributions $f_\Pi(V_\Pi|V_\Delta)$ and $f_\Delta(V_\Delta|V_\Pi)$, we shall make use of cumulative distributions, denoted by symbol F, e.g.,

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[19]This assumption should not be taken too literally. It simply means that the public has incomplete information, and that a probability distribution is being used to represent their beliefs.

[20]This is what is often meant by *first-order beliefs*. More precisely, first-order beliefs correspond to the uncorrelated case, i.e. $f_\Pi(V_\Pi)$ and $f_\Delta(V_\Delta)$ being separately known. The correlated j.p.d. cannot be described by first-order beliefs alone due to the fact that e.g. defendant's belief about the distribution of plaintiff's valuations depends on his own valuation, and, therefore is not known to others. The correlated example is the simplest one to go beyond first-order beliefs; and yet it renders calculations analytically tractable.

$$F_\Delta(V_\Delta|V_\Pi) = \int_0^{V_\Delta} f_\Delta(v|V_\Pi)\,dv. \tag{II.3}$$

Now, each player is faced with the problem of making the optimal bid, given his private valuation of the asset. In doing so, he must have some knowledge of other player's intentions, i.e., other player's strategy. Therefore, we should end up with a pair of coupled equations for the strategies. Also note that in contrast to games with complete information, in which the strategy is essentially a number (the actual bid), under incomplete information the *strategy* should be defined as a function that maps players' internal valuations into bids: $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$[21]. We shall, in fact, work with the inverses of these functions, viz., $V_\Pi(s)$ and $V_\Delta(s)$[22]. We shall also use the latter to define the distributions $\widehat{f}_\Pi(s) \equiv f_\Pi(V_\Pi(s))\,V'_\Pi(s)$ and $\widehat{f}_\Delta(s) \equiv f_\Delta(V_\Delta(s))\,V'_\Delta(s)$, which are then the probability distributions of the actual bids. We also define the corresponding conditional probability distributions $\widehat{f}_\Pi(s|V_\Delta)$ and $\widehat{f}_\Delta(s|V_\Pi)$ and their cumulative versions, $\widehat{F}_\Pi(s|V_\Delta)$ and $\widehat{F}_\Delta(s|V_\Pi)$.

If Π's internal valuation is V_Π, his bid $s_\Pi(V_\Pi)$ is determined so as to maximize his payoff. In this maximization he assumes that Δ follows his optimal strategy $s_\Delta(V_\Delta)$. As Π does not know Δ's private valuation V_Δ, he maximizes his expected payoff under the condition that V_Δ is randomly distributed according to $f_\Delta(V_\Delta|V_\Pi)$. Then Δ's private valuation does not influence Π's payoff, except through Δ's bid. Equivalently, Π works with the distribution of Δ's bids, $\widehat{f}_\Delta(s|V_\Pi)$.

We now proceed to implement the determination of optimal strategies outlined in the previous paragraph. After the plaintiff and the defendant have submitted their secret bids, s_Π and s_Δ, the asset goes the higher bidder at damages determined by the lower bidder. Therefore, the plaintiff's and the defendant's respective payoffs, π_Π and π_Δ, are given by

$$\pi_\Pi(s_\Pi, s_\Delta|V_\Pi, V_\Delta) = \begin{cases} V_\Pi - D_\Pi(s_\Delta), & \text{if } s_\Pi > s_\Delta, \\ D_\Delta(s_\Pi), & \text{if } s_\Pi < s_\Delta, \end{cases} \tag{II.4a}$$

$$\pi_\Delta(s_\Pi, s_\Delta|V_\Pi, V_\Delta) = \begin{cases} D_\Pi(s_\Delta), & \text{if } s_\Pi > s_\Delta, \\ V_\Delta - D_\Delta(s_\Pi), & \text{if } s_\Pi < s_\Delta. \end{cases} \tag{II.4b}$$

Taking into account the probabilities of private valuations, we can write down the expressions for the expected payoffs:

$$\pi_\Pi(s_\Pi|V_\Pi) = \int_0^{s_\Pi} ds_\Delta\,[V_\Pi - D_\Pi(s_\Delta)]\widehat{f}_\Delta(s_\Delta|V_\Pi) + D_\Delta(s_\Pi)[1 - \widehat{F}_\Delta(s_\Pi|V_\Pi)], \tag{II.5a}$$

$$\pi_\Delta(s_\Delta|V_\Delta) = \int_0^{s_\Delta} ds_\Pi\,[V_\Delta - D_\Delta(s_\Pi)]\widehat{f}_\Pi(s_\Pi|V_\Delta) + D_\Pi(s_\Delta)[1 - \widehat{F}_\Pi(s_\Delta|V_\Delta)]. \tag{II.5b}$$

[21] We thus ignore the possibility of mixed strategies.

[22] The inverses exist as long as $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$ are monotonic, and we shall assume that they are.

Here, $\pi_\Pi(s_\Pi|V_\Pi)$ has the meaning of Π's expected payoff, given that his valuation is V_Π and his bid is s_Π; and similarly for $\pi_\Delta(s_\Delta|V_\Delta)$. The players follow their optimal strategies, i.e., they maximize their individual payoffs with respect to their individual bids. Thus, the bids obey the stationarity condition:

$$\frac{d}{ds_\Pi}\pi_\Pi(s_\Pi|V_\Pi) = \frac{d}{ds_\Delta}\pi_\Delta(s_\Delta|V_\Delta) = 0. \tag{II.6}$$

Inserting the explicit expressions for the expected payoffs Eqs. (II.5a) and (II.5b) leads to the conditions:

$$[V_\Pi(s_\Pi) - D_\Pi(s_\Pi)]\widehat{f}_\Delta(s_\Pi|V_\Pi) + D'_\Delta(s_\Pi)[1 - \widehat{F}_\Delta(s_\Pi|V_\Pi)] - D_\Delta(s_\Pi)\widehat{f}_\Delta(s_\Pi|V_\Pi) \quad = \quad 0, \tag{II.7a}$$

$$[V_\Delta(s_\Delta) - D_\Delta(s_\Delta)]\widehat{f}_\Pi(s_\Delta|V_\Delta) + D'_\Pi(s_\Delta)[1 - \widehat{F}_\Pi(s_\Delta|V_\Delta)] - D_\Pi(s_\Pi)\widehat{f}_\Pi(s_\Delta|V_\Delta) \quad = \quad 0, \tag{II.7b}$$

which, upon rearrangement, may be written as the following set of conditions:

$$\begin{aligned} D'_\Pi(s) &= \lambda_\Pi(s|V_\Delta(s))[D_\Pi(s) + D_\Delta(s) - V_\Delta(s)], && \text{(II.8a)} \\ D'_\Delta(s) &= \lambda_\Delta(s|V_\Pi(s))[D_\Pi(s) + D_\Delta(s) - V_\Pi(s)], && \text{(II.8b)} \end{aligned}$$

the coefficients λ_Π and λ_Δ in the conditions being defined via

$$\begin{aligned} \lambda_\Pi(s|V_\Delta) &\equiv \widehat{f}_\Pi(s|V_\Delta)\Big/[1 - \widehat{F}_\Pi(s|V_\Delta)], && \text{(II.8c)} \\ \lambda_\Delta(s|V_\Pi) &\equiv \widehat{f}_\Delta(s|V_\Pi)\Big/[1 - \widehat{F}_\Delta(s|V_\Pi)]. && \text{(II.8d)} \end{aligned}$$

In principle, conditions (II.8c) and (II.8d) should, for a given pair of damages functions $D_\Pi(s)$ and $D_\Delta(s)$, be solved for the (inverse) bidding strategy functions. This appears to be a formidable task, as the equations are, in general, non-linear [the non-linearity coming from the dependence of λ_Π and λ_Δ on $V_\Pi(s)$ and $V_\Delta(s)$]. However, for reasons that we shall explain in section III, a less demanding route is available and, in fact, appropriate.

2. Continuous put rule

We now examine the continuous put rule. The only difference from the continuous call case is that the damages are determined by the winning bid. Accordingly, we have the expected payoffs:

$$\pi_\Pi(s_\Pi|V_\Pi) = [V_\Pi - D_\Pi(s_\Pi)]\widehat{F}_\Delta(s_\Pi|V_\Pi) + \int_{s_\Pi}^{\infty} ds_\Delta\, D_\Delta(s_\Delta)\widehat{f}_\Delta(s_\Delta|V_\Pi), \tag{II.9a}$$

$$\pi_\Delta(s_\Delta|V_\Delta) = [V_\Delta - D_\Delta(s_\Delta)]\widehat{F}_\Pi(s_\Delta|V_\Delta) + \int_{s_\Delta}^{\infty} ds_\Pi\, D_\Pi(s_\Pi)\widehat{f}_\Pi(s_\Pi|V_\Delta), \tag{II.9b}$$

and, again, construct stationarity conditions: $(d/ds_\Pi)\pi_\Pi(s_\Pi|V_\Pi) = (d/ds_\Delta)\pi_\Delta(s_\Delta|V_\Delta) = 0$. By substituting the explicit expressions for the expected payoffs, stationarity conditions can be written in the following form:

$$-D'_\Pi(s_\Pi)\,\widehat{F}_\Delta(s_\Pi|V_\Pi) + [V_\Pi(s_\Pi) - D_\Pi(s_\Pi) - D_\Delta(s_\Pi)]\widehat{f}_\Delta(s_\Pi|V_\Pi) = 0, \tag{II.10a}$$

$$-D'_\Delta(s_\Delta)\,\widehat{F}_\Pi(s_\Delta|V_\Delta) + [V_\Delta(s_\Delta) - D_\Delta(s_\Delta) - D_\Delta(s_\Delta)]\widehat{f}_\Pi(s_\Delta|V_\Delta) = 0. \tag{II.10b}$$

We can recast this conditions into a form similar to that of Eqs. (II.8a) and (II.8b),

$$D'_\Pi(s) = \mu_\Delta(s|V_\Pi(s))[V_\Pi(s) - D_\Pi(s) - D_\Delta(s)], \tag{II.11a}$$

$$D'_\Pi(s) = \mu_\Pi(s|V_\Delta(s))[V_\Delta(s) - D_\Pi(s) - D_\Delta(s)], \tag{II.11b}$$

where we have introduced

$$\mu_\Pi(s|V_\Delta) \equiv \widehat{f}_\Pi(s|V_\Delta)\Big/\widehat{F}_\Pi(s|V_\Delta), \tag{II.11c}$$

$$\mu_\Delta(s|V_\Pi) \equiv \widehat{f}_\Delta(s|V_\Pi)\Big/\widehat{F}_\Delta(s|V_\Pi). \tag{II.11d}$$

D. Recovering the discrete rules

We have seen that when the asset-allocation decision is delegated to the litigants the economic efficiency of the allocation is increased by a suitable choice of the damages in a vanilla call or put regime. Work on exotic liability rules has been sparked by the sense that efficiency would be further increased if the litigants were to have the greater freedom offered by iterated call or put regimes, and the court were to have at its disposal the correspondingly greater number of adjustable damages parameters—infinitely many in the continuous call and put cases.

Let us put these arguments on a firmer basis. It has been argued that the vanilla call is, in general, more efficient than the property rule, as the property rule can be thought of as vanilla call with damages D set to infinity. Barring the exceptional possibility that the total efficiency is independent of D, a liability call rule can always be made more efficient than a property rule by a better choice of D. As we shall now explain, the continuous call rule is more efficient than the iterated call, provided that the iterated call can be shown to follow from the continuous version with appropriately chosen parameters. This is indeed possible if the damages curve of the continuous version is chosen to have a zigzag shape, as shown in figure 2.

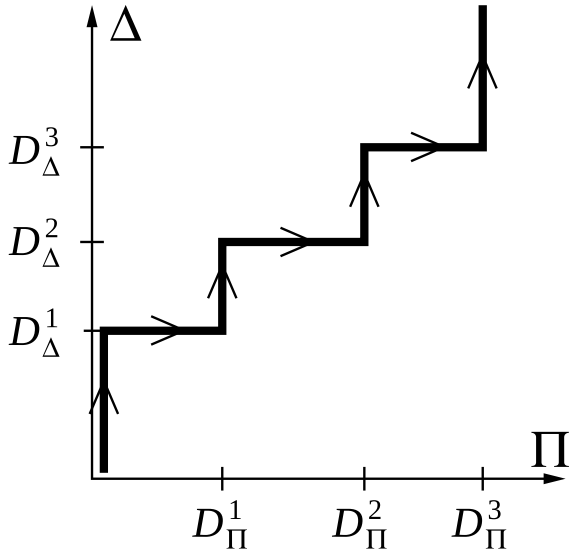

Fig. 2.— A continuous rule bidding curve that simulates a 3-stage iterated call rule.

With this zigzag shaped damages curve, the plaintiff should only bid at corners at which the zigzag curve turns right, and the defendant should only bid at corners where the curve turns left (or at the point at the origin, or infinitely far away). To be precise, we will show that if one of the players would use the strategies from this subgame (i.e. bidding only in corners), the other player gains no advantage by deviating from the subgame, i.e. by bidding away from corners. Indeed, if the defendant decides to place a bid somewhere on a horizontal segment, he would simply increase the amount of damages he receives if he loses by moving his bid point to the right. If he decides to bid somewhere on a vertical segment, he is indifferent where to place his bid, but it is safer to move down, thereby decreasing the damages he might have to pay (without changing the damages he might receive), should the plaintiff decide to play away from corner. This shows the sought reduction from the continuous call to the discrete iterated call. Similar reasoning can be used to relate the continuous put and the discrete iterated put rules. We refer the reader to the A for a table of damage functions used to simulate each of the liability rules described in the Introduction.

E. Connections with auctions

The procedure we have described in this chapter has much in common with familiar auctions. In the case of auctions, the relative of our iterated call is the increasing price, or English, auction. The iterated put, in turn, corresponds to the decreasing price, or Dutch, auction. Arguments similar to the ones we have made (in going from the infinitesimal bid increments to "sealed envelope" bidding) led Vickrey[23] to propose replacing English auctions with the so-called Vickrey, or second-best, auction, and to replace the Dutch auction with the first-best auction. In both Vickrey and first-best auctions, the bidders secretly submit their bids. The highest bid becomes the winner at the price determined by the second-highest (Vickrey) or the highest (first-best) bid. The analogy ends here, because in our formulation, the damages paid, do not, in general, correspond to the real bids. The importance of the Vickrey auction is in that the bidders bid according to their private valuations, whereas in first-best auctions, they tend to strategically underbid. In contrast, in our scheme, the bid amounts under the iterated put rule reflects strategic underbidding, whilst strategic overbidding features under the iterated call rule.

III. Designing Optimal Mechanism

A. Formulation

The task of designing an optimal mechanism can be split into two parts. The first part involves finding optimal bidding strategies given a set of damage parameters. The second part involves maximizing total expected efficiency by adjusting these parameters. For the continuous call or put rules the strategy is synonymous with submitting a bid, and the set of parameters corresponds to the damages functions $D_\Pi(s)$ and $D_\Delta(s)$. The problem of determining the optimal bid, given $D_\Pi(s)$ and $D_\Delta(s)$, was addressed and solved (at least formally) in the previous section. Approaching the second part of the problem (i.e. the maximization of total expected utility), we note that for the *first-best allocation* (i.e. a mechanism whereby the party with the higher private valuation always gains control of the asset) the total expected efficiency achieves its optimal value. As we shall now see, a model in which the set of parameters corresponds to the pair of continuous functions is sufficiently rich to achieve first-best allocational efficiency. Any other mechanism will be, at best, only as efficient, not more so.

Under both the continuous call and the continuous put rules, the asset is allocated to the party with the higher bid, i.e. to the plaintiff if $s_\Pi > s_\Delta$ and to the defendant if $s_\Pi < s_\Delta$. On the other hand, for optimal allocation we must have that the asset is allocated to the plaintiff if $V_\Pi(s_\Pi) > V_\Delta(s_\Delta)$ and to the defendant if $V_\Pi(s_\Pi) < V_\Delta(s_\Delta)$. A moment's reflection will show that these conditions are compatible if and only if $V_\Pi(s) = V_\Delta(s)$. On the other hand, we observed that the game is invariant[24] with respect to the reparametrization of $D_\Pi(s)$ and $D_\Delta(s)$ via any monotonically increasing function $t(s)$. Assuming that there exists a mechanism [i.e. functions

[23] William Vickrey, *Counterspeculation, Auctions and Competitive Sealed Tenders,* 16 J. Finance 8 (1961).

[24] To be precise, the term *covariant* should be used, as the calculated bids should be adjusted according to the reparametrization.

$D_\Pi(s)$ and $D_\Delta(s)$] that guarantee $V_\Pi(s) = V_\Delta(s)$, by choosing $t = V_\Pi(s) = V_\Delta(s)$ as the new parameter, the mechanism can be reformulated via $D_\Pi(t)$ and $D_\Delta(t)$ so that $V_\Pi(t) = V_\Delta(t) = t$.

We now substitute $V_\Pi(s) = s$ and $V_\Delta(s) = s$ into the equations Eqs. (II.8a) and (II.8b) of section C for the optimal bids so that $D_\Pi(s)$ and $D_\Delta(s)$ are viewed as the unknowns rather than parameters:

Call	*Put*
$D'_\Pi(s) = \lambda_\Pi(s\|V_\Delta = s)[D_\Pi(s) + D_\Delta(s) - s]$	$D'_\Pi(s) = \mu_\Delta(s\|V_\Pi = s)[s - D_\Pi(s) - D_\Delta(s)]$
$D'_\Delta(s) = \lambda_\Delta(s\|V_\Pi = s)[D_\Pi(s) + D_\Delta(s) - s]$	$D'_\Delta(s) = \mu_\Pi(s\|V_\Delta = s)[s - D_\Pi(s) - D_\Delta(s)]$

Table 2: Differential equations for damages functions.

Litigants, furnished with $D_\Pi(s)$, $D_\Delta(s)$ that solve these equations, would apply the rationale of section C, and in so doing would discover that their optimal strategies are $V_\Pi(s) = V_\Delta(s) = s$. Therefore, in making their bids, they would be forced to reveal their private valuations. By virtue of the fact that the asset is allocated to the higher bidder, first-best allocational efficiency is realized.

A special note should be made about the boundary conditions obeyed by the damages functions. To ensure that the equations of Table 2 have non-singular solutions, the following conditions must be met[25]:

Call	*Put*
$D_\Pi(s_{\max}) + D_\Delta(s_{\max}) = s_{\max}$	$D_\Pi(s_{\min}) + D_\Delta(s_{\min}) = s_{\min}$

Table 3: Boundary conditions for damages functions.

We now address the task of actually solving for the damages functions, given the optimal strategies $V_\Pi(s) = V_\Delta(s) = s$. To do this, we first introduce the auxiliary function $\widetilde{D}(s) \equiv D_\Pi(s) + D_\Delta(s)$. As seen by adding together the differential equations of Table 2, $\widetilde{D}(s)$ obeys a certain ordinary differential equation, depending on whether we are considering the continuous call rule or the continuous put rule:

Call	*Put*
$\widetilde{D}'(s) = \lambda(s)[\widetilde{D}(s) - s]$	$\widetilde{D}'(s) = \mu(s)[s - \widetilde{D}(s)]$
$\widetilde{D}(s_{\max}) = s_{\max}$	$\widetilde{D}(s_{\min}) = s_{\min}$

Table 4: Equations for $\widetilde{D}(s)$ together with boundary conditions.

where $\lambda(s) \equiv \lambda_\Pi(s|V_\Delta = s) + \lambda_\Delta(s|V_\Pi = s)$ and $\mu(s) \equiv \mu_\Pi(s|V_\Delta = s) + \mu_\Delta(s|V_\Pi = s)$; the quantities $\lambda_\Pi, \lambda_\Delta, \mu_\Pi, \mu_\Delta$ have been defined in section C. The stated boundary conditions on $\widetilde{D}$

[25] We introduce the following notation: $s_{\max}$ is the smallest s such that $f(t_\Pi, t_\Delta) = 0$ for all $t_\Pi, t_\Delta > s$; similarly, $s_{\min}$ is the largest s such that $f(t_\Pi, t_\Delta) = 0$ for all $t_\Pi, t_\Delta < s$.

follow from those given above for $D_\Pi(s)$ and $D_\Delta(s)$. Note that, owing to the linear, first-order form of the equation obeyed by $\widetilde{D}$, the method of integrating factors presents us with an explicit solution in terms of the functions $\lambda(s)$ or $\mu(s)$, which encode information about the joint probability distribution $f(V_\Pi, V_\Delta)$. Note that this distribution may be arbitrarily correlated.

Armed with the solution for $\widetilde{D}(s)$, we may return to one or other of the differential equations, for $D_\Pi(s)$ or for $D_\Delta(s)$, eliminate the sum $D_\Pi(s) + D_\Delta(s)$ on the right hand side in favor of $\widetilde{D}(s)$, and integrate to obtain $D_\Pi(s)$ and $D_\Delta(s)$.

Note, that $D_\Pi(s)$ and $D_\Delta(s)$ are determined only up to a single constant of integration, as they should be. If a pair $\{D_\Pi(s), D_\Delta(s)\}$ is a solution, then so is the pair $\{D_\Pi(s) + A, D_\Delta(s) - A\}$.

As in Ayres & Goldbart (20xx) it is possible for courts to decouple distributional and allocative concerns in that there is a family of allocatively equivalent damage curves that vary how the total expected value is divided between the plaintiff and defendant. In what follows, this constant of integration can be thought of as a free variable that a lawmaker can set to independently pursue equitable goals or to enhance ex ante investment incentives.

B. Examples

In this section we shall consider some elementary but, we hope, instructive examples involving the continuous call and continuous put rules. We shall take the joint probability distribution density to be constant (i.e. uniform distribution) throughout some geometrical region in the (V_Π, V_Δ) plane. We remind the reader that a rectangular region having sides parallel to the coordinate axes corresponds to an example of uncorrelated distribution[26], whereas an arbitrary shape implies correlations.

Uniform distributions lead to particularly simple expressions for the functions $\lambda_\Pi(s)$ and $\lambda_\Delta(s)$, as we shall now see. We refer to figure 3 for further discussion of this fact. There, the point S corresponds to $V_\Pi = V_\Delta = s$. Points, where the edge of the area is intersected by the upward and rightward rays drawn from S are labeled U and R, respectively. Then the functions $\lambda_\Pi(s) = f(s,s) \Big/ \int_s^\infty f(t,s)dt$ and $\lambda_\Delta(s) = f(s,s) \Big/ \int_s^\infty f(s,t)dt$ assume the following simple forms: $\lambda_\Pi(s) = 1/|SR|$ and $\lambda_\Delta(s) = 1/|SU|$, where $|AB|$ denotes the distance between points A and B. Similarly, for the functions $\mu_\Pi(s)$ and $\mu_\Delta(s)$ that appear in the continuous put case, we obtain (see figure 4): $\mu_\Pi(s) = 1/|SL|$ and $\mu_\Delta(s) = 1/|SB|$. We now use these results to compute damages functions in various example settings.

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[26]Indeed, for the rectangular region $f(V_\Pi, V_\Delta) = f_\Pi(V_\Pi) f_\Delta(V_\Delta)$, with $f_\Pi(V_\Pi) = 1/(V_\Pi^{\max} - V_\Pi^{\min})$ for $V_\Pi \in [V_\Pi^{\min}; V_\Pi^{\max}]$ and $f_\Delta(V_\Delta) = 1/(V_\Delta^{\max} - V_\Delta^{\min})$ for $V_\Delta \in [V_\Delta^{\min}; V_\Delta^{\max}]$, is constant everywhere in the rectangle having corners $(V_\Pi^{\min}, V_\Delta^{\min})$, $(V_\Pi^{\min}, V_\Delta^{\max})$, $(V_\Pi^{\max}, V_\Delta^{\max})$, $(V_\Pi^{\max}, V_\Delta^{\min})$ and 0 elsewhere.

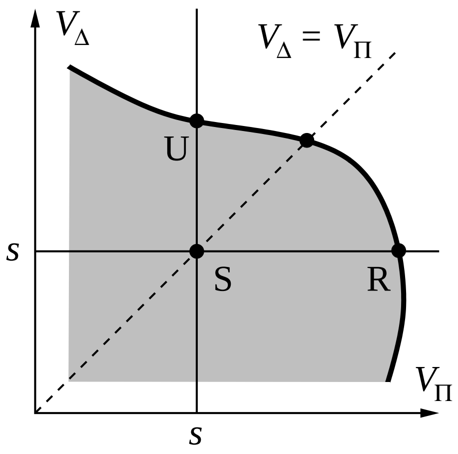

Fig. 3.— Determining $\lambda_\Pi(s)$ and $\lambda_\Delta(s)$ for a uniform distribution.

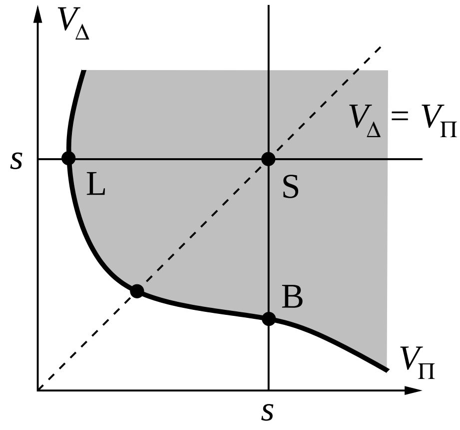

Fig. 4.— Determining $\mu_\Pi(s)$ and $\mu_\Delta(s)$ for a uniform distribution.

1. Uncorrelated uniform distribution: continuous call rule

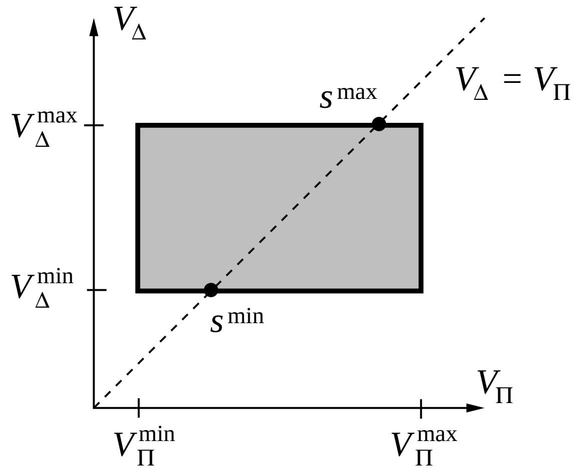

Fig. 5.— Uniform distribution in a rectangle.

Let us take the joint probability distribution to be uniform in the rectangle $V_\Pi \in [V_\Pi^{\min}; V_\Pi^{\max}]$, $V_\Delta \in [V_\Delta^{\min}; V_\Delta^{\max}]$. For this case, we have $\lambda_\Pi(s) = 1/(V_\Pi^{\max} - s)$ and $\lambda_\Delta(s) = 1/(V_\Delta^{\max} - s)$. For the sake of concreteness, let us further assume that $V_\Pi^{\max} > V_\Delta^{\max}$. In this case, the boundary condition (see section A) is enforced at $s_{\max} = V_\Delta^{\max}$. To solve for $\widetilde{D}(s)$, we multiply the equation it obeys by the integrating factor $(V_\Pi^{\max} - s)(V_\Delta^{\max} - s)$, thus obtaining

$$(V_\Pi^{\max} - s)(V_\Delta^{\max} - s)\widetilde{D}'(s) = (V_\Pi^{\max} + V_\Delta^{\max} - 2s)(\widetilde{D}(s) - s). \tag{III.1}$$

Then, transferring the term proportional to $\widetilde{D}(s)$ to the left hand side and integrating gives

$$(V_\Pi^{\max} - s)(V_\Delta^{\max} - s)\widetilde{D}(s) = \frac{2}{3}s^3 - \frac{1}{2}(V_\Pi^{\max} + V_\Delta^{\max})s^2 + K, \tag{III.2}$$

where the constant of integration K is fixed by the boundary condition that $\widetilde{D}(s)$ remain finite at $s = V_\Delta^{\max}$. At that value of s, both sides of the equation must vanish. Straightforward algebra then yields the following expression for $\widetilde{D}(s)$:

$$\widetilde{D}(s) = \frac{1}{6}(V_\Pi^{\max} + V_\Delta^{\max}) + \frac{2}{3}s - \frac{1}{6}\frac{(V_\Pi^{\max} - V_\Delta^{\max})^2}{V_\Pi^{\max} - s}, \tag{III.3}$$

which we remind the reader holds for the $V_\Pi^{\max} > V_\Delta^{\max}$ case of the uniform rectangular distribution. Inserting this solution for $\widetilde{D}(s)$ into the equation obeyed by $D_\Pi(s)$ gives

$$D'_\Pi(s) = \frac{1}{3} - \frac{1}{6}\frac{V_\Pi^{\max} - V_\Delta^{\max}}{V_\Pi^{\max} - s} - \frac{1}{6}\frac{(V_\Pi^{\max} - V_\Delta^{\max})^2}{(V_\Pi^{\max} - s)^2}, \tag{III.4}$$

which may be straightforwardly integrated. Thus, one obtains $D_\Pi(s)$ and, using the relation $D_\Delta(s) = \widetilde{D}(s) - D_\Pi(s)$, also $D_\Delta(s)$:

$$\begin{aligned} D_\Pi(s) &= \frac{1}{3}s + \frac{1}{6}(V_\Pi^{\max} - V_\Delta^{\max})\ln(V_\Pi^{\max} - s) - \frac{1}{6}\frac{(V_\Pi^{\max} - V_\Delta^{\max})^2}{(V_\Pi^{\max} - s)} + A, && \text{(III.5a)} \\ D_\Delta(s) &= \frac{1}{3}s - \frac{1}{6}(V_\Pi^{\max} - V_\Delta^{\max})\ln(V_\Pi^{\max} - s) + \frac{1}{6}(V_\Pi^{\max} + V_\Delta^{\max}) - A, && \text{(III.5b)} \end{aligned}$$

where A is the constant of integration. The resulting damages curve is shown in figure 6.

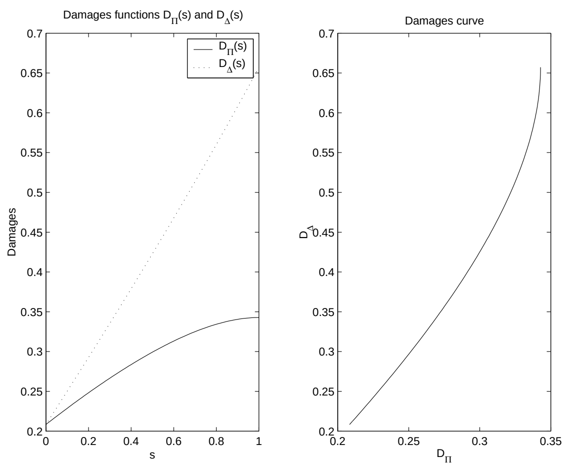

Fig. 6.— Damages obtained by considering the distribution uniform in a rectangle $[0; 2] \times [0; 1]$. Left: Damages curve in the $D_\Pi - D_\Delta$ plane. Right: Actual damages functions $D_\Pi(s)$ and $D_\Delta(s)$.

The strategy just used also provides damages curves for the case $V_\Pi^{\max} < V_\Delta^{\max}$, in which case we obtain

$$
\begin{aligned}
D_\Pi(s) &= \frac{1}{3}s + \frac{1}{6}(V_\Delta^{\max} - V_\Pi^{\max})\ln(V_\Delta^{\max} - s) - \frac{1}{6}\frac{(V_\Delta^{\max} - V_\Pi^{\max})^2}{(V_\Delta^{\max} - s)} + A, &\text{(III.6a)}\\
D_\Delta(s) &= \frac{1}{3}s - \frac{1}{6}(V_\Delta^{\max} - V_\Pi^{\max})\ln(V_\Delta^{\max} - s) + \frac{1}{6}(V_\Delta^{\max} + V_\Pi^{\max}) - A, &\text{(III.6b)}
\end{aligned}
$$

as well as for the case $V_\Pi^{\max} = V_\Delta^{\max}$, for which we obtain

$$
D_{\Pi,\Delta}(s) = \frac{1}{6}V^{\max} + \frac{1}{3}s \pm A\,. \tag{III.7}
$$

We note that especially simple damage curves (i.e. they obey $D_\Pi - D_\Delta = \text{const}$) result for symmetrical problems, by which we mean settings for which $\lambda_\Pi(s) = \lambda_\Delta(s)$ [or, in the put-rule case, $\mu_\Pi(s) = \mu_\Delta(s)$]. These situations commonly arise for probability distributions that are symmetric with respect to the interchange of players: $f(V_\Pi, V_\Delta) = f(V_\Delta, V_\Pi)$. Also note that setting $V_\Pi^{\min} = V_\Delta^{\min} = 0$ and $V_\Pi^{\max} = V_\Delta^{\max} = 1$, would correspond to the special case addressed by Ayres and Balkin.[27]

2. *Uncorrelated uniform distribution: continuous put rule*

Let us consider the same geometry as we did in the previous example, i.e., a uniform rectangular distribution. Now, however, for the sake of concreteness, let us assume that $V_\Pi^{\min} < V_\Delta^{\min}$, in which case the appropriate boundary condition is enforced at $s_{\min} = V_\Delta^{\min}$. Again we multiply the equation of Table 4 by a suitable integrating factor, thus obtaining

$$
(s - V_\Pi^{\min})(s - V_\Delta^{\min})\widetilde{D}'(s) = (2s - V_\Pi^{\min} - V_\Delta^{\min})[s - \widetilde{D}(s)]. \tag{III.8}
$$

Integrating and applying the boundary condition gives

$$
\widetilde{D}(s) = \frac{1}{6}(V_\Delta^{\min} + V_\Pi^{\min}) + \frac{2}{3}s + \frac{1}{6}\frac{(V_\Delta^{\min} - V_\Pi^{\min})^2}{s - V_\Pi^{\min}}. \tag{III.9}
$$

Then, substituting this result into the equation for $D_\Delta(s)$ gives

$$
D'_\Delta(s) = \frac{1}{s - V_\Pi^{\min}}\left[\frac{1}{3}s - \frac{1}{6}(V_\Delta^{\min} + V_\Pi^{\min}) - \frac{1}{6}\frac{(V_\Delta^{\min} - V_\Pi^{\min})^2}{s - V_\Pi^{\min}}\right], \tag{III.10}
$$

and integrating and using $D_\Pi(s) = \widetilde{D}(s) - D_\Delta(s)$ gives the damages functions:

---

[27] See *supra* note 6.

$$D_\Pi(s) = \frac{1}{3}s + \frac{1}{6}(V_\Delta^{\rm min} - V_\Pi^{\rm min})\ln(s - V_\Pi^{\rm max}) + \frac{1}{6}(V_\Delta^{\rm min} + V_\Pi^{\rm min}) + A, \qquad \text{(III.11a)}$$

$$D_\Delta(s) = \frac{1}{3}s - \frac{1}{6}(V_\Delta^{\rm min} - V_\Pi^{\rm min})\ln(s - V_\Pi^{\rm max}) + \frac{1}{6}\frac{(V_\Delta^{\rm min} - V_\Pi^{\rm min})^2}{s - V_\Pi^{\rm min}} - A, \qquad \text{(III.11b)}$$

where A is the constant of integration. These particular results hold for the $V_\Pi^{\rm min} < V_\Delta^{\rm min}$ case of the uniform rectangular distribution. Similar results can readily be obtained for the cases $V_\Pi^{\rm min} = V_\Delta^{\rm min}$ and $V_\Pi^{\rm min} > V_\Delta^{\rm min}$.

3. A simple correlated distribution: continuous call rule

The purpose of the present exercise is to exhibit an example in which the valuations are correlated but, nevertheless, the damages curve may be explicitly obtained.

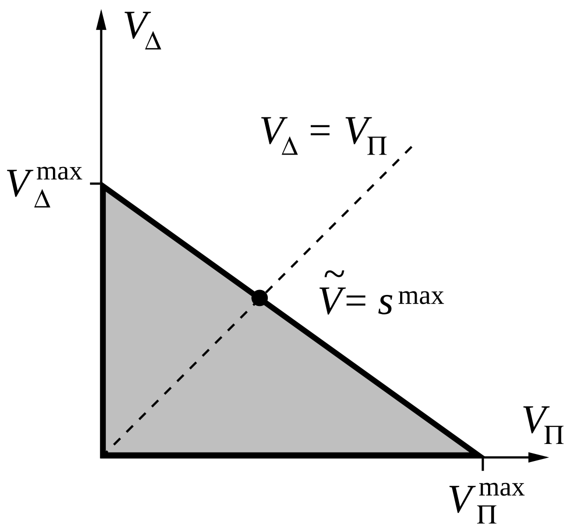

Fig. 7.— Uniform distribution in a triangle.

Here we take the joint probability distribution to be uniform in the equilateral triangle with corners $(0,0)$, $(V_\Pi^{\rm max}, 0)$ and $(0, V_\Delta^{\rm max})$. Alternatively, we view this region as the first quadrant ($V_\Pi > 0, V_\Delta > 0$) bounded by the line $(V_\Pi/V_\Pi^{\rm max}) + (V_\Delta/V_\Delta^{\rm max}) = 1$. The intersection with the line $V_\Delta = V_\Pi$ determines $s_{\rm max} = \widetilde{V} \equiv V_\Pi^{\rm max}V_\Delta^{\rm max}/(V_\Pi^{\rm max} + V_\Delta^{\rm max})$. For this distribution it is straightforward to show that $\lambda_\Pi(s) = \lambda(s)V_\Pi^{\rm max}/(V_\Pi^{\rm max} + V_\Delta^{\rm max})$ and $\lambda_\Delta(s) = \lambda(s)V_\Delta^{\rm max}/(V_\Pi^{\rm max} + V_\Delta^{\rm max})$, where $\lambda(s) \equiv 1/(\widetilde{V} - s)$. Consequently, the equation of Table 4 for $\widetilde{D}(s)$ takes on the simple form

$$\widetilde{D}'(s) = \frac{\widetilde{D}(s) - s}{\widetilde{V} - s}, \qquad \text{(III.12)}$$

which can immediately be rewritten as

$$\frac{d}{ds}\left[(\widetilde{V}-s)\widetilde{D}(s)\right] = -s, \tag{III.13}$$

and hence integrated to give, upon applying the boundary condition $\widetilde{D}(\widetilde{V}) = \widetilde{V}$, the result

$$\widetilde{D}(s) = (\widetilde{V}+s)/2. \tag{III.14}$$

Substituting this result into the differential equation for $D_\Pi(s)$ gives

$$D'_\Pi(s) = \frac{V_\Pi^{\max}}{V_\Pi^{\max} - V_\Delta^{\max}} \frac{(\widetilde{V}+s)/2 - s}{\widetilde{V} - s} = \frac{1}{2}\frac{V_\Pi^{\max}}{V_\Pi^{\max} - V_\Delta^{\max}}. \tag{III.15}$$

Integrating and using $D_\Delta(s) = \widetilde{D}(s) - D_\Pi(s)$ then gives the damages functions

$$D_\Pi(s) = \frac{1}{2}\frac{V_\Pi^{\max}}{V_\Pi^{\max} + V_\Delta^{\max}} s + A, \tag{III.16a}$$

$$D_\Delta(s) = \frac{1}{2}\frac{V_\Delta^{\max}}{V_\Pi^{\max} + V_\Delta^{\max}} s + \widetilde{V} - A, \tag{III.16b}$$

where A is, again, the constant of integration; these results can be expressed in the more symmetrical form

$$D_\Pi(s) = \frac{V_\Pi^{\max}}{2}\frac{V_\Delta^{\max} + s}{V_\Pi^{\max} + V_\Delta^{\max}} + A, \tag{III.17a}$$

$$D_\Delta(s) = \frac{V_\Delta^{\max}}{2}\frac{V_\Pi^{\max} + s}{V_\Pi^{\max} + V_\Delta^{\max}} - A. \tag{III.17b}$$

4. Uniform triangular distribution: continuous put rule

This example turns out to be the least interesting of the four we have chosen. As long as the lower left corner of the region in which the distribution is nonzero has the form of the rectangle, the expressions for $\lambda_\Pi(s)$ and $\lambda_\Delta(s)$ are the same as for the rectangle (with $V_\Pi^{\min} = V_\Delta^{\min} = 0$ in this example). Hence, the damages functions are identical to those found for the rectangle. To rephrase this more sharply, the damages functions do not depend on any probability weight that lies outside the box $V_\Pi < s_{\max}$, $V_\Delta < s_{\max}$ (for the continuous put rule), or the box $V_\Pi > s_{\min}$, $V_\Delta > s_{\min}$ (for the continuous call rule).

For completeness of the solution we do need to specify the damages for $s > s^{\max}$. In fact the appropriate choice is to set $D_\Pi(s) = D_\Pi(s^{\max})$, $D_\Delta(s) = D_\Delta(s^{\max})$ for $s > s^{\max}$.

5. General solution: continuous call rule

In this section we shall derive explicit formulas for the damages functions which can be used for arbitrary distributions of valuations, correlated or otherwise. As we did for the examples, we first solve for $\widetilde{D}(s)$, which can be seen from the equations in Table 4 to obey the differential equation

$$\widetilde{D}'(s) = \lambda(s)[\widetilde{D}(s) - s] \tag{III.18}$$

together with the boundary condition $\widetilde{D}(s_{\max}) = s_{\max}$. As the differential equation is a first-order ordinary one, we use the method of integrating factors, by which we find that

$$\frac{d}{ds}\left[\widetilde{D}(s)\, e^{\int_s^{s_{\max}} \lambda(u)du}\right] = -s\lambda(s)\, e^{\int_s^{s_{\max}} \lambda(u)du}, \tag{III.19}$$

and hence that

$$\widetilde{D}(s)\, e^{\int_s^{s_{\max}} \lambda(u)du} = K + \int_s^{s_{\max}} dt\, t\, \lambda(t)\, e^{\int_t^{s_{\max}} \lambda(u)du}, \tag{III.20}$$

where K is the constant of integration.

Next, we determine K by applying the boundary condition. To do this, we consider the limit $s \to s_{\max}$, bearing in mind that $\lambda(s)$ is singular in this limit. By making use of the elementary identity[28]

$$e^{\int_s^{s_{\max}} \lambda(u)du} = \int_s^{s_{\max}} dt\, \lambda(t)\, e^{\int_t^{s_{\max}} \lambda(u)du}, \tag{III.21}$$

and observing that, in the limit $s \to s_{\max}$, we may replace t on the right hand side of the solution for $\widetilde{D}(s)$ by $s_{\max}$, we recognize that the boundary condition is satisfied if $K = 0$. Thus, we arrive at the solution

$$\widetilde{D}(s) = \int_s^{s_{\max}} dt\, t\, \lambda(t)\, e^{-\int_s^t \lambda(u)du}. \tag{III.22}$$

To complete our task, we insert the formula for $D(s)$ into the differential equation obeyed by $D_\Pi(s)$ [we could equally well have used $D_\Delta(s)$] to obtain

$$D'_\Pi(s) = \lambda_\Pi(s) \int_s^{s_{\max}} dt\, (t - s)\lambda(t)\, e^{-\int_s^t \lambda(u)du}, \tag{III.23}$$

where $\lambda_\Pi(s)$ is given in section A; integrating yields the result:

[28] Obtained by observing that the r.h.s. is the integral of a total derivative.

$$D_\Pi(s) = A + \int_{s_{\min}}^{s} dt\, \lambda_\Pi(t) \int_t^{s_{\max}} dt'\, (t'-t)\lambda(t')\, e^{-\int_s^{t'} \lambda(u)du}\,. \tag{III.24}$$

As we now know $\widetilde{D}(s)$ and $D_\Pi(s)$, it is straightforward to construct $D_\Delta(s)$, using $D_\Pi(s) + D_\Delta(s) = \widetilde{D}(s)$. As mentioned at the start of this section, these damages formulas can be used for any joint probability distributions of valuations, the latter featuring through the quantities $\lambda_\Pi(s)$ and $\lambda_\Delta(s)$; see section A.

6. General solution: continuous put rule

We devote this section to deriving the general solution for the continuous put rule. As only slight modifications of the formalism for the continuous put rule are needed, the explanations will be brief. As we know from Table 4, the combined damages $\widetilde{D}(s)$ satisfy

$$\widetilde{D}'(s) = \mu(s)[s - \widetilde{D}(s)], \tag{III.25}$$

together with the boundary condition $D(s_{\min}) = s_{\min}$. Solving this equation together with the boundary conditions by the method of integrating factors yields the following result:

$$\widetilde{D}(s) = \int_{s_{\min}}^{s} dt\, t\mu(t) e^{-\int_t^s \mu(u)du}. \tag{III.26}$$

Closer examination reveals that this form also satisfies the boundary condition at $s \to s_{\min}$. Inserting the result into the differential equation for $D_\Pi(s)$ from Table 2 and integrating, we obtain:

$$D_\Pi(s) = A + \int_{s_{\min}}^{s} dt\, \mu_\Delta(t) \int_{s_{\min}}^{t} dt'\, (t-t')\, \lambda(t')\, e^{-\int_{t'}^{t} \mu(u)du}. \tag{III.27}$$

The defendant's damages $D_\Delta(s)$ are obtained using $D_\Delta(s) = \widetilde{D}(s) - D_\Pi(s)$.

IV. Game theoretic formulation

In passing to the continuum limit of the iterated call and put rules, we have designed the damages functions in such a way that the plaintiff's and defendant's optimal strategies become $V_\Pi(s) = V_\Delta(s) = s$, i.e., each reveals his private information. The idea that incentive problems can be efficiently solved by designing a mechanism under which rational participants reveal their private information truthfully has become known as the *revelation principle* and was originally formulated by J. Mirrlees[29]. As we now discuss, one can use the revelation principle to construct a general formulation of the problem of efficient asset allocation in the context of liability rules.

We present a view of liability rules as *games* between the plaintiff and the defendant played according to rules stipulated by the court. Each player makes a *move* (for instance, announces a number that identifies one of his possible strategies), accounting for his private valuation and the common information at his disposal. The players move at the same time, and the court chooses the final asset holder and the damages exchanged (e.g. by looking them up in a table having rows and columns corresponding to the plaintiff's and defendant's moves). In this strategic form, the players, who have full knowledge of the payoff table, solve the problem of finding their optimal strategies, and make their moves accordingly. However, they could equivalently delegate their decision-making to the court by revealing their private valuation, provided they are assured that the court will use reasoning identical to theirs and will make the corresponding moves on their behalf. It must be stressed that in making a decision on the plaintiff's behalf the court should pretend that it does not know the defendant's private valuation, and vice versa, in order to correctly mimic the litigants' behavior. The next step consists of combining the two steps—finding the optimal moves and determining the winner (who will become the owner of the asset) and the damages—into one. The court asks the litigants to submit their private information and uses these valuations to determine who is to be the asset holder and the damages. The added requirement is that the mechanism be *incentive compatible*[30]—the litigants must not be able to gain any advantage by misrepresenting their private information.

With this scheme in mind, if we are aiming at achieving the perfect efficiency, the court should allocate the asset to the party with the higher private valuation. If the announced valuations are s_Π and s_Δ, then the asset should go to the plaintiff if $s_\Pi > s_\Delta$ and to the defendant if $s_\Pi < s_\Delta$ (or to either party if $s_\Pi = s_\Delta$). Furthermore, the court sets the damages at $D(s_\Pi, s_\Delta)$; the function D has to be crafted in an incentive-compatible way. Note that the function $D(s_\Pi, s_\Delta)$ will, in general, be discontinuous across $s_\Pi = s_\Delta$. It will, therefore, be convenient to work with two functions, $D_\Pi(s_\Pi, s_\Delta)$ and $D_\Delta(s_\Pi, s_\Delta)$, defined only for $s_\Pi \geq s_\Delta$ and $s_\Pi \leq s_\Delta$, respectively.

To find the necessary restrictions placed on $D_\Pi(s_\Pi, s_\Delta)$ and $D_\Delta(s_\Pi, s_\Delta)$, let us assume that the players have the private valuations V_Π and V_Δ, and that their optimal strategies $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$ are not necessarily revealing. (They are revealing if $s_\Pi(V_\Pi) = V_\Pi$ and $s_\Delta(V_\Delta) = V_\Delta$.) We shall then enforce the condition that if any one's strategy is revealing, the opponent's best response is to follow a revealing strategy. In other words, we shall seek damages functions such that this scenario is always realized.

As usual, we choose $f(V_\Pi, V_\Delta)$ to denote the joint probability distribution governing the valu-

[29] James Mirrlees, *An Exploration in the Theory of Optimal Income Taxation*, Rev. Econ. Stud. (1971), 15 J. Legal Stud. 93 (1986)

[30] See, e.g., *infra* note 36.

ations. For convenience, we use the Heaviside function $\theta(x)$[31] as a tool for restricting the regions of integration. For instance, we would multiply the integrand by $\theta(V_\Pi - V_\Delta)$ to restrict the region of integration to the half-plane $V_\Pi > V_\Delta$. We also make use of θ's formal derivative (known as the Dirac δ-function) $\theta'(x) \equiv \delta(x)$. The latter has a meaning only inside an integral, so that $\int dx\, \delta(x-a)\, f(x) = f(a)$. We regard the litigants' expected payoffs, π_Π and π_Δ, as entities that depend on the strategy functions $s_\Pi(V_\Pi)$ and $s_\Delta(V_\Delta)$[32]. In the present context, in which the asset goes to the higher bidder, the expected payoff functionals are given by

$$\begin{aligned}
\pi_\Pi[s_\Pi(\cdot), s_\Delta(\cdot)] &= \int\!\!\int dV_\Pi\, dV_\Delta\, f(V_\Pi, V_\Delta)\, \{\theta(s_\Pi - s_\Delta)[V_\Pi - D_\Pi(s_\Pi, s_\Delta)] \\
&\qquad + \theta(s_\Delta - s_\Pi) D_\Delta(s_\Pi, s_\Delta)\}\,, && \text{(IV.1)} \\
\pi_\Delta[s_\Pi(\cdot), s_\Delta(\cdot)] &= \int\!\!\int dV_\Pi\, dV_\Delta\, f(V_\Pi, V_\Delta)\, \{\theta(s_\Pi - s_\Delta) D_\Pi(s_\Pi, s_\Delta) \\
&\qquad + \theta(s_\Delta - s_\Pi)[V_\Delta - D_\Delta(s_\Pi, s_\Delta)]\}\,. && \text{(IV.2)}
\end{aligned}$$

The restriction on the damages, which we are seeking, is chosen so that the revealing strategies $s_\Pi(V_\Pi) = V_\Pi$ and $s_\Delta(V_\Delta) = V_\Delta$ make the litigants' expected payoff stationary with respect to variations in strategy:

$$\frac{\delta}{\delta s_\Pi(V_\Pi)} \pi_\Pi[s_\Pi(\cdot), s_\Delta(\cdot)] = \frac{\delta}{\delta s_\Delta(V_\Delta)} \pi_\Delta[s_\Pi(\cdot), s_\Delta(\cdot)] = 0. \qquad \text{(IV.3)}$$

Note that we are using *functional derivatives,*[33] rather than conventional ones. Functional derivatives can be thought of as partial derivatives in a multidimensional space of infinitely many variables [i.e. $s_1 = s(V_1), s_2 = s(V_2), \ldots$, where $V_1, V_2, \ldots$ enumerate all real values]. In the simple case in which $\pi[s(V)]$ is expressible in the form

$$\pi[s(\cdot)] = \int dV\, p(s(V)), \qquad \text{(IV.4)}$$

the functional derivative turns out to be computable via[34]:

$$\frac{\delta\pi[s(\cdot)]}{\delta s(V)} = \left.\frac{dp}{ds}\right|_{s=s(V)}. \qquad \text{(IV.5)}$$

[31] The Heaviside function may be defined via $\theta(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$.

[32] It is customary to refer to entities that depend on functions, rather than, say, variables, as *functionals* of those functions.

[33] See, e.g., C. Nash, *Relativistic Quantum Fields,* (Academic Press, London, 1978).

[34] Note that the functional derivative of a scalar is itself a function (of V).

The stationarity conditions Eq. (IV.3) are, indeed, expressible in this form.

Intuitively, the condition $\delta\pi_\Pi/\delta s_\Pi(V_\Pi) = 0$ is solved by first fixing V_Π and then optimizing the payoff by adjusting s_Π. The procedure, repeated for all V_Π, yields the function equation

$$0 = \int dV_\Delta\, f(V_\Pi, V_\Delta)\Big\{ - \theta(s_\Pi - s_\Delta) D_\Pi^{(\Pi)}(s_\Pi, s_\Delta) + \theta(s_\Delta - s_\Pi) D_\Delta^{(\Pi)}(s_\Pi, s_\Delta) + \delta(s_\Pi - s_\Delta)[V_\Pi - D_\Pi(s_\Pi, s_\Delta) - D_\Delta(s_\Pi, s_\Delta)]\Big\}, \tag{IV.6a}$$

where $D^{(\Pi)}(s_\Pi, s_\Delta)$ is used to denote the partial derivative $\partial D(s_\Pi, s_\Delta)/\partial s_\Pi$. Similarly, optimizing the defendant's payoff we obtain

$$0 = \int dV_\Pi\, f(V_\Pi, V_\Delta)\Big\{\theta(s_\Pi - s_\Delta) D_\Pi^{(\Delta)}(s_\Pi, s_\Delta) - \theta(s_\Delta - s_\Pi) D_\Delta^{(\Delta)}(s_\Pi, s_\Delta) + \delta(s_\Pi - s_\Delta)[V_\Delta - D_\Pi(s_\Pi, s_\Delta) - D_\Delta(s_\Pi, s_\Delta)]\Big\}, \tag{IV.6b}$$

where $D^{(\Delta)}(s_\Pi, s_\Delta)$ denotes the partial derivative $\partial D(s_\Pi, s_\Delta)/\partial s_\Delta$. The next step is to determine the constraints on D_Π and D_Δ imposed by the demand that revealing strategies are an equilibrium. Thus, we insert $s_\Pi(V_\Pi) = V_\Pi$ and $s_\Delta(V_\Delta) = V_\Delta$ into Eqs. (IV.6a) and (IV.6b), arriving at the conditions

$$\begin{aligned} 0 &= \int dV_\Delta\, f(V_\Pi, V_\Delta)\Big\{ - \theta(V_\Pi - V_\Delta) D_\Pi^{(\Pi)}(V_\Pi, V_\Delta) + \theta(V_\Delta - V_\Pi) D_\Delta^{(\Pi)}(V_\Pi, V_\Delta)\Big\} \\ &\quad + f(V_\Pi, V_\Pi)\Big\{V_\Pi - D_\Pi(V_\Pi, V_\Pi) - D_\Delta(V_\Pi, V_\Pi)\Big\}, && \text{(IV.7a)} \\ 0 &= \int dV_\Pi\, f(V_\Pi, V_\Delta)\Big\{\theta(V_\Pi - V_\Delta) D_\Pi^{(\Delta)}(V_\Pi, V_\Delta) - \theta(V_\Delta - V_\Pi) D_\Delta^{(\Delta)}(V_\Pi, V_\Delta)\Big\} \\ &\quad + f(V_\Delta, V_\Delta)\Big\{V_\Delta - D_\Pi(V_\Delta, V_\Delta) - D_\Delta(V_\Delta, V_\Delta)\Big\}. && \text{(IV.7b)} \end{aligned}$$

Recall that section III was devoted to determining the conditions obeyed by the damages functions for the continuous call and continuous put rules. We can recover these conditions from the more general structure, just developed. To see this, recall that, under the continuous call rule, the damages were determined by the losing bid. In the present language, this reads

$$D_\Pi(V_\Pi, V_\Delta) = D_\Pi(V_\Delta), \tag{IV.8a}$$

$$D_\Delta(V_\Pi, V_\Delta) = D_\Delta(V_\Pi). \tag{IV.8b}$$

Hence, not only do $D_\Pi^{(\Pi)} = D_\Delta^{(\Delta)} = 0$, but also $D_\Pi^{(\Delta)}$ and $D_\Delta^{(\Pi)}$ can be taken out of the integrations in Eqs. (IV.7a) and (IV.7b), which then become

$$0 = D'_\Delta(V_\Pi)\int dV_\Delta\, f(V_\Pi,V_\Delta)\theta(V_\Delta - V_\Pi) + f(V_\Pi,V_\Pi)[V_\Pi - D_\Pi(V_\Pi) - D_\Delta(V_\Pi)], \tag{IV.9a}$$

$$0 = D'_\Pi(V_\Delta)\int dV_\Pi\, f(V_\Pi,V_\Delta)\theta(V_\Pi - V_\Delta) + f(V_\Delta,V_\Delta)[V_\Delta - D_\Pi(V_\Delta) - D_\Delta(V_\Delta)], \tag{IV.9b}$$

i.e., precisely the conditions on $D_\Pi(V_\Delta)$ and $D_\Delta(V_\Pi)$ obtained in section III. Similarly, for the case of the continuous put rule, by setting $D_\Pi(V_\Pi,V_\Delta) = D_\Pi(V_\Pi)$ and $D_\Delta(V_\Pi,V_\Delta) = D_\Delta(V_\Delta)$ we recover precisely the conditions on $D_\Pi(V_\Pi)$ and $D_\Delta(V_\Delta)$ obtained in section III. To conclude this section, let us make a few remarks about the validity of these results. We have proven the stationarity of the revealing strategies but have not demonstrated that they constitute maxima of the expected payoffs. Thus, the conditions we have found are necessary, but not (necessarily) sufficient for the existence of the revealing Nash equilibrium. In B, we shall prove that these conditions are also sufficient for the uncorrelated case: $f(V_\Pi,V_\Delta) = f_\Pi(V_\Pi)\, f_\Delta(V_\Delta)$. The uniqueness of the Nash equilibrium remains an open problem even for the uncorrelated case.

V. Concluding remarks

In this Paper we have proposed two new types of liability rule—the continuous call and the continuous put. In contrast to traditional liability rules, which depend on a single damage parameter D, the continuous rules are specified by the functions $D_\Pi(s)$ and $D_\Delta(s)$. These rules can be thought of as the infinite-stage limit of the iterated call and iterated put rule or, equivalently, as an entirely new procedure, akin to the Vickrey and first-best auctions. We have shown that these rules are rich, in the sense that (a) all rules encountered, to date, can be expressed as special cases of either the continuous call or the continuous put rule, and (b) by appropriately choosing $D_\Pi(s)$ and $D_\Delta(s)$, it is possible to design a mechanism capable of achieving the first-best efficiency, assuming that the plaintiff and the defendant are both rational players. We have also reformulated the problem using the revelation principle, and have obtained the conditions satisfied by the damages function $D(V_\Pi, V_\Delta)$; we have also shown how the continuous call and put rules manifest themselves as two special cases of this most general mechanism.

The striking similarities between our results for the continuous call and continuous put rules can be explained as a manifestation of the *duality* of these two mechanisms. Indeed, if we assume the existence of an upper bound on private valuations, i.e., the price M such that, with probability 1, both $V_\Pi, V_\Delta \le M$. [That $V_\Pi, V_\Delta \ge 0$ is implicit.] To each asset, we may assign the corresponding liability—a contract to acquire the asset for the amount M. For bilateral disputes, the right to be free from such liability represents the asset dual to the original one.[35] It can be argued that the outcome of the continuous call rule is equivalent to the outcome of the continuous put for dual assets, and the outcome of the continuous put rule for the original assets is the same at that of the continuous call for the dual assets. Thus, one might say that the distinction between the two is only superficial. It remains to be seen whether any of the more general damages functions that satisfy the efficiency conditions of section IV correspond to rules of any practical value.

A famous (negative) result, due to Myerson and Satterthwaite,[36] states that it is impossible to have an efficient trading mechanism. To be precise, whenever domains of non-zero probability overlap, no mechanism is possible that is both (a) individually rational [for players to participate in the procedure] and (b) first-best efficient. Work by Chatterjee and Samuelson[37] relaxes the second constraint in order to solve the problem for the case of a distribution that is uniform in the unit $[0;1] \times [0;1]$ square. The present Paper relaxes the first constraint.[38] While courts can assure that the disputants' expected payoffs (given the court's knowledge of the general value probability distributions) are positive, the payoffs of privately informed playoffs can be negative. But this potential participation problem is equally acute for traditional liability rules.[39]

Thus, in future work, it would be useful to analyze the extent to which a participation constraint limits the analysis. For now, it is at least clear that litigation often extinguishes the participation choice of one disputant. Defendants, by polluting (or by seeking a declaratory), can often force plaintiff participation in the mechanism. And plaintiffs, by suing, might often force defendant participation. The possible need to assure the participation of only one relevant disputant might mean that it may be appropriate for courts to divide the total payoffs (through its setting of A) in a way that assures that the appropriate range of disputes is subjected to the continuous call or continuous put mechanism.

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[35]This feature is reminiscent of the call-put parity result for vanilla liability rule. See *supra* note 7 and *supra* note 10.

[36]R.B. Myerson and M.A. Satterthwaite, *Efficient Mechanisms for Bilateral Trading*, J. of Econ. Theory 29, 265 (1983).

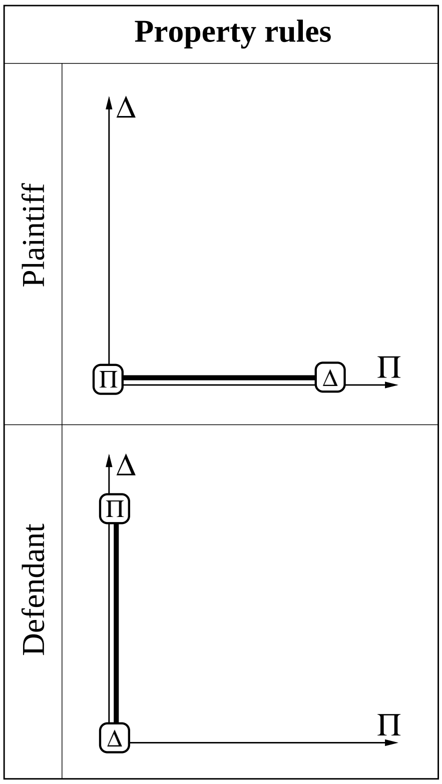
[37]K. Chatterjee and W. Samuelson, *Bargaining under Incomplete Information*, 31 Oper. Research 835, 837-38 (1983).

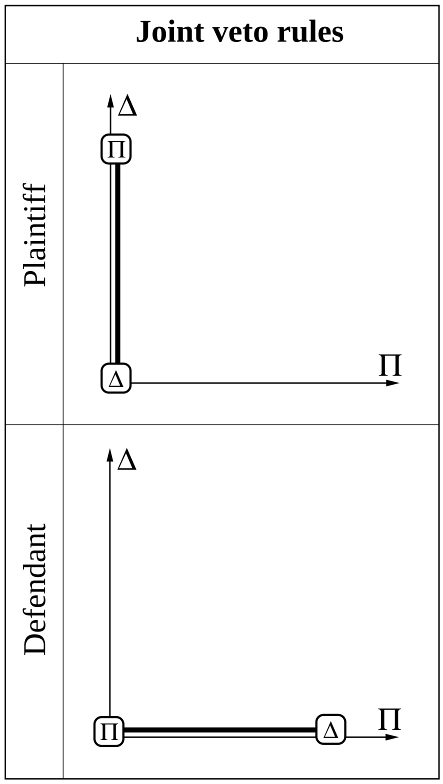
[38]Peter Cramton, Robert Gibbons, and Paul Klemperer, in *Dissolving a Partnership Efficiently* [Econometrica 55, 615 (1987)], formulate a set of conditions under which a partnership (i.e. three or more players dividing two or more assets) can efficiently be dissolved. They restrict their attention to symmetric, uncorrelated distributions, and formulate additional constraints on the distributions that make efficient allocation possible.

[39]Saul Levmore, *Unifying Remedies: Property Rules, Liability Rules, and Startling Rules*, 106 Yale L. J. 2149 (1997).

A. Damage curves table

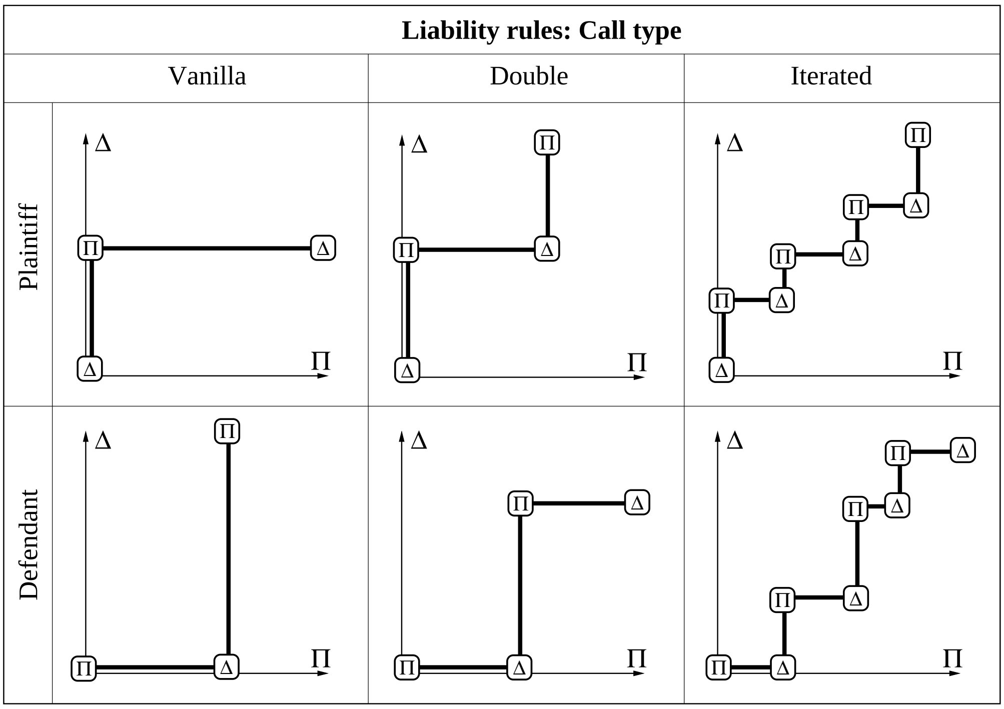
As we have shown in section II, the continuous call and continuous put rules (with appropriately chosen damages functions) can be used to effectively simulate any of the discrete liability rules mentioned in the Introduction. In this appendix, we provide a complete list of damages curves for all sixteen rules. The rows in all tables correspond to the initial entitlement holder. The continuous call is used throughout, except when simulating put options, for which the continuous put should be used. Circles with letters Π and Δ label points where the plaintiff and the defendant, respectively, might place their bids.





B. Demonstration of Nash equilibrium for the uncorrelated case

The purpose of this appendix is to show that, provided that the distribution of valuations is uncorrelated, and provided that the damages functions obey the conditions given in Eqs. (IV.7a) and (IV.7b), the strategy in which both players reveal their private information is a Nash equilibrium. To see this, consider the equations obeyed by the damages functions for the uncorrelated case:



$$0 = \int dV_\Delta \, f_\Delta(V_\Delta) \left\{ -\theta(V_\Pi - V_\Delta)\, D_\Pi^{(\Pi)}(V_\Pi, V_\Delta) + \theta(V_\Delta - V_\Pi)\, D_\Delta^{(\Pi)}(V_\Pi, V_\Delta) \right\} + f_\Delta(V_\Pi)\, [V_\Pi - D_\Pi(V_\Pi, V_\Pi) - D_\Delta(V_\Pi, V_\Pi)], \tag{B1}$$

$$0 = \int dV_\Pi \, f_\Pi(V_\Pi) \left\{ \theta(V_\Pi - V_\Delta)\, D_\Pi^{(\Delta)}(V_\Pi, V_\Delta) - \theta(V_\Delta - V_\Pi)\, D_\Delta^{(\Delta)}(V_\Pi, V_\Delta) \right\} + f_\Pi(V_\Delta)\, [V_\Delta - D_\Pi(V_\Delta, V_\Delta) - D_\Delta(V_\Delta, V_\Delta)]. \tag{B2}$$

Assume that the plaintiff's valuation is V_Π. The plaintiff's payoff from making the bid s_Π can, provided that the defendant uses the revealing strategy $s_\Delta = V_\Delta$, be expressed as

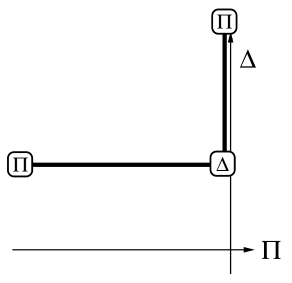
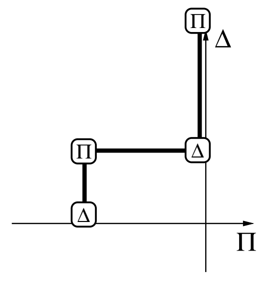
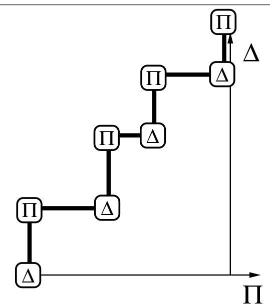
$$\pi_\Pi[s_\Pi | V_\Pi] = \int dV_\Delta \, f_\Delta(V_\Delta) \{ \theta(s_\Pi - V_\Delta)[V_\Pi - D_\Pi(s_\Pi, V_\Delta)] + \theta(V_\Delta - s_\Pi)\, D_\Delta(s_\Pi, V_\Delta) \}. \tag{B3}$$

Now, we can safely assume that s_Π is such that $f_\Pi(s_\Pi) > 0$.[40] Then, by differentiating the plaintiff's payoff $\pi_\Pi[s_\Pi | V_\Pi]$ with respect to s_Π, and comparing the result with Eq. (IV.7a) for damages (having substituted $v_\Pi = s_\Pi$ in the latter), we can see that

[40] For instance, the court can assess a huge fine [setting $D_\Pi(s_\Pi, v_\Delta) = +\infty$ or $D_\Delta(s_\Pi, v_\Delta) = -\infty$] on a player (plaintiff) who reports an impossible private valuation.

$$\frac{d}{ds_\Pi}\pi_\Pi[s_\Pi|V_\Pi] = f_\Delta(s_\Pi)\,(V_\Pi - s_\Pi), \tag{B4}$$

i.e., $d\pi_\Pi/ds_\Pi > 0$ when $s_\Pi < V_\Pi$ and $d\pi_\Pi/ds_\Pi < 0$ when $s_\Pi > V_\Pi$. This proves that $s_\Pi = V_\Pi$ is the plaintiff's best response. This analysis, applied to the defendant's payoff, shows that $s_\Delta = V_\Delta$ is the defendant's best response, provided the plaintiff uses the revealing strategy. Therefore, the revealing strategies do, indeed, form a Nash equilibrium.

Liability rules: Put type			
	Vanilla	Double	Iterated
Plaintiff			
Defendant